SOLUTION OF COEFFICIENT INVERSE PROBLEMS OF HEAT CONDUCTION WITH ACCOUNT OF A PRIORI INFORMATION ON THE VALUES OF THE SOUGHT FUNCTIONS

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UDC 536.24

An algorithm is proposed to numerically solve coefficient inverse problems of heat conduction accounting for inequalities and equalities imposed on the sought functions.

The efficiency of computational algorithms for solving incorrect inverse problems of heat transfer is largely dependent on the possibility of taking into account the available a priori information on the sought functions [1, 2]. Taking, as a case in point, the solution of a coefficient inverse problem of heat conduction, the current study analyzes iterational computational algorithms allowing for such a priori known properties of the temperature-dependent sought functions as their positiveness and specified values of these functions at certain values of the argument. The consideration of a priori information of this kind necessitates the numerical solution of extremal problems with imposed inequalities or equalities.

We examine a one-dimensional heat transfer process, whose mathematical model has the form of a boundary-value problem for a homogeneous quasi-linear heat conduction equation

$$C(T)\frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial T}{\partial x}\right), \quad 0 < x < l, \quad 0 < \tau \leqslant \tau_m; \tag{1}$$

$$T(x, 0) = T_0(x), \quad 0 \leqslant x \leqslant l; \tag{2}$$

$$\alpha_1 \lambda(T(0, \tau))\frac{\partial T(0, \tau)}{\partial x} + \beta_1 T(0, \tau) = q_1(\tau); \tag{3}$$

$$\alpha_2 \lambda(T(b, \tau))\frac{\partial T(l, \tau)}{\partial x} + \beta_2 T(l, \tau) = q_2(\tau), \tag{4}$$

where α_1, β_1, α_2, and β_2 are parameters, which can be used to analyze boundary conditions of the first, second, or third kind. Let, at a certain number N of spatial points with coordinates $x = X_i$, $i = \overline{1, N}$, the time dependences of the temperature be measured

$$T_{\text{mea}}(X_i, \tau) = f_i(\tau), \quad i = \overline{1, N}. \tag{5}$$

The coefficient inverse problem of heat conduction consists of determining the functions C(T) or/and λ(T) from conditions (1)-(5). It is assumed in this case that the type of boundary conditions (3) and (4) as well as the number of heat sensors N satisfy the conditions assuring uniqueness of the analyzed inverse problem (see, for example, [3, 4]).

For simplicity of the subsequent presentation we consider the case of determining one characteristic λ(T). Extending the analyzed algorithms to multiparametric inverse problems does not present any considerable difficulties, although it involves more cumbersome calculations.

In constructing the iterational algorithms to solve the coefficient inverse problems, wherein the sought characteristics are functions of the temperature, parametrization of unknown functions must be employed. For example, the temperature dependence of the thermal conductivity may be written approximately in the form

Moscow Aviation Institute. Translated from Inzhenerno-fizicheskii Zhurnal, Vol. 64, No. 1, pp. 113-119, January, 1993. Original article submitted January 9, 1992.

 1062-0125/93/6401-0092$12.50

$$\lambda(T) = \sum_{k=1}^{m} p_k \varphi_k(T) \equiv \langle \bar{p}, \bar{\varphi} \rangle, \quad T \in [a\ b], \tag{6}$$

where $\bar{p} = [p_1, p_2, \ldots, p_m]^T$ is the parametric vector; $p \in R^m$; R^m is the m-dimensional Euclidean space; $\bar{\varphi} = [\varphi_1(T), \varphi_2(T), \ldots, \varphi_m(T)]^T$ is the vector of the given basic functions; $\langle \cdot, \cdot \rangle$ is the scalar product; and a and b are the minimal and the maximal values of the temperature, respectively.

The mathematical model (1)-(4) at the prescribed characteristic allows computation of the temperature at the points where the heat sensors are installed in accordance with the measurement procedure (5). Thereby $\lambda(T)$ is transformed into the vector function $f = \{f_i(\tau), i = \overline{1, N}\}$ resulting from the measurements. In approximation (7), the parametric vector $\bar{p}$ is sought. In this case, using the condition of equality of the computed and measured temperatures, the inverse problem may be presented as an operator equation

$$A\bar{p} = f, \quad \bar{p} \in R^m, \quad f \in F, \tag{7}$$

where the operator A is constructed on the basis of the model (1)-(4) taking the measurement procedure (5) into account; and F is the space of the measured functions, for which the space L_2 of the functions with an integrable square is generally utilized.

The characteristics sought and, hence, the basic functions $\varphi_k(T)$, $k = \overline{1, m}$, in the parametric presentation of the form (6) must satisfy definite smoothness requirements. These requirements ensue from the conditions of differentiability of a residual functional [2]. For example, the temperature dependence of the thermal conductivity $\lambda(T)$ must be a twice continuously differentiable function. Continuity of the first derivative is essential for the volumetric specific heat C(T). Therefore, the basic functions $\varphi_k(T)$, $k = \overline{1, m}$, should be chosen with regard to the indicated requirements. The smoothness requirements are fulfilled, in particular, by cubic B-splines [5].

A salient feature of the inverse problems of heat transfer is their incorrectness. This feature most often manifests itself in the fact that minor errors on the right side of Eq. (7) may lead to great deviations in the solution. On conversion to the finite-dimensional approximation (6) the inverse operator A^{-1} in Eq. (7) becomes bounded but the inverse problem remains poorly conditioned. Special regularizing methods and algorithms [6] are needed to solve the incorrect inverse problems.

An iteration regularization method [2] proved to be highly efficient in solving various inverse problems of heat transfer. In this method an iteration sequence, minimizing the residual functional, is constructed to solve the inverse problem of the form (7)

$$J(p) = \|A\bar{p} - f\|_F^2 = \sum_{i=1}^{N} \int_0^{\tau_m} [T(X_i, \tau) - f_i(\tau)]^2 d\tau. \tag{8}$$

Here, use is made of gradient methods of first-order optimization. Successive approximations are constructed by the formula

$$\bar{p}^{s+1} = \bar{p}^s + \gamma_s \bar{G}(\bar{J}'_p{}^{(s)}), \quad s = 0, 1, \ldots, s^*, \tag{9}$$

where s is the iteration number, γ_1 is the descent parameter, $\bar{J}_p'$ is the gradient of the functional (2.1.11) calculated in the space R^m, $G(J')$ is the vector characterizing the employed optimization method, p^0 is the initial approximation specified a priori, and s^* is the number of the last iteration determined during the solution of the problem from the regularizing residual condition

$$J(\bar{p}) \simeq \delta^2 \tag{10}$$

(δ^2 is the prescribed measurement error computed in the metric of the space F). The descent parameter γ_s is obtained from the condition

$$\gamma_s = \operatorname*{Arg\,min}_{\gamma_s > 0} J(\bar{p}^s + \gamma \bar{G}(\bar{J}_p'^{(s)})) \tag{11}$$

by any familiar method, for example, "golden section" [7]. The gradient $\bar{J}'_p$ is calculated using the solution of a boundary-value problem for a conjugate variable [2].

The algorithms to solve the coefficient inverse problems must be constructed by taking into account limitations ensuing from physical considerations

$$\lambda(T) > 0, \quad T \in [a, b]. \tag{12}$$

However, it is generally assumed that, in the course of minimizing the residual functional (8), the solution satisfies all possible restrictions, and unconditional optimization methods are used.

The practice of solving the coefficient inverse problems indicates that, during the iterational refinement of the sought characteristics, condition (12) may be violated. This leads to instability of the direct problem, and the solution of the analyzed inverse problem is unfeasible. In such a situation, another initial approximation to the sought functions is usually chosen, and computations are repeated. This technique, however, may also lead to the violation of constraints (12). As a result, the solution of the inverse problem turns into a fairly arduous investigation, in which the researcher's intuition and subjectivity reveal themselves actively.

To eliminate the above-mentioned factors and to construct universal computational algorithms for solving inverse problems, minimizing the residual functional (8), conditional optimization methods should be utilized, for example, a method of projection of conjugate gradients. In this case, successive approximations are constructed by the formula

$$\bar{p}^{s+1} = P_W(\bar{p}^s + \gamma \bar{G}(\bar{J}_p^{\prime(s)})), \tag{13}$$

where P_W is the projection operator on a set of admissible solutions W, which is constructed with consideration of constraints (12).

A projection $\bar{w}_0$ of the assigned vector $\bar{p}$ on the set $W \subset R^m$ is determined from the condition [7]

$$\|\bar{p} - \bar{w}_0\|^2 = \inf_{\bar{w} \in W} \psi(\bar{w}), \quad \psi(\bar{w}) = \|\bar{p} - \bar{w}\|^2, \tag{14}$$

where $\|\cdot\|$ is the norm in the space R^m. In the extremal problem (14), it is necessary to formulate conditions under which the vector w is an element of the set W. Taking account of constraints (12), we present W as a closed set

$$W = \left\{ u(T) = \langle \bar{w}, \bar{\varphi} \rangle = \sum_{k=1}^{m} w_k \varphi_k(T) \geq \bar{\alpha}, \quad T \in [a, b] \right\}, \tag{15}$$

here $\bar{\alpha} > 0$ is the prescribed quantity and $\varphi_k(T)$, $k = \overline{1, m}$, are the basic functions of the approximation. Constraints on the vector $\bar{w}$ are formulated proceeding from the fact that condition (15) is fulfilled on each interval of the approximation. However, such an approach results in a system of nonlinear constraints, and the solution of the extremal problem (14) necessitates the employment of appropriate methods of conditional optimization [8], which are extremely complicated and laborious, as far as computation is concerned.

A considerable simplification of the algorithm of projecting on the set of admissible solutions may be provided by introducing a system of linear constraints on the vector $\bar{w}$. To construct such an approximate algorithm, we introduce on the interval [a, b] the net

$$w = \{T_i = a + (i-1)h, \quad i = 1, \ldots, n, \quad h = (b-a)(n-1), \; n > m\}, \tag{16}$$

where n is the prescribed number. At each node of the net (16), we require that condition (15) be fulfilled. As a result, we obtain the system of linear constraints

$$\sum_{k=1}^{m} w_k \varphi_k(T_i) = \bar{\alpha}, \quad i = \overline{1, n}. \tag{17}$$

In matrix notation, constraints (17) may be represented as

$$D\bar{w} = \bar{\alpha}, \tag{18}$$

where D is an $m \times n$ matrix, whose elements are equal to $d_{k,i} = \varphi_k(T_i)$, and α is an assigned vector of dimension $n > m$, wherein $\alpha_i = \bar{\alpha}$, $i = \overline{1, n}$. Then the extremal problem (14) converts to the form

$$\psi(\bar{w}_0) = \inf_{D\bar{w} \geqslant \bar{\alpha}} \|\bar{p} - \bar{w}\|^2. \tag{19}$$

Denoting further $\bar{p} - \bar{w} = \bar{v}$, $C = -D$, and $\bar{\beta} = \bar{\alpha} = D\bar{p}$ we obtain

$$\psi(\bar{v}_0) = \inf_{C\bar{v} \geqslant \bar{\beta}} \|\bar{v}\|^2. \tag{20}$$

Thus, the construction of the projection on a set of admissible solutions in the approximate formulation is reduced to the search for a vector of the minimal norm under linear inequalities. This is a well studied problem, and effective computational algorithms [9] have been worked out to solve it.

For a practical realization of the outlined projection algorithm it is necessary to specify the parameters of the additional grid (16), in particular, the number of nodes n. In this algorithm, the solution of the extremal problem (14) using a least-square method is ensured by the fulfillment of the relationship $n > m$. The computational experiments performed show that for solution of the problem (20), it is usually sufficient to satisfy the condition $n > (5\text{-}10)\ m$.

The quality of the approximate solution of incorrect inverse problems is dependent in many respects on the completeness of taking into account the entire available a priori information on the sought characteristics. The allowance for such information narrows a set of admissible solutions for the inverse problem and, as a consequence, improves certainty and accuracy of the results [1, 2].

As the a priori information, we may prescribe, specifically, values of the sought function $\lambda(T)$ at certain temperatures

$$\lambda(T_l) = \mu_l, \quad l = \overline{1, L}. \tag{21}$$

For example, in determining the thermal conductivity of high-temperature materials, the value of this characteristic at room temperature $\lambda(T_0) = \mu_0$ may be known.

The consideration of a priori information in the form of the constraints-equalities of the type (21) lies in eliminating a part of the unknown parameters from the approximating relation (6). As a result, the dimension of the sought vector is reduced, and the system of basic functions is modified. The restrictions on the vector of coefficients in expression (6) are written in the form of a system of linear algebraic equations

$$\sum_{k=1}^{m} p_k \varphi_k(T_l) = \mu_l, \quad l = \overline{1, L} \tag{22}$$

whence, for example, using the Gauss elimination method with the choice of the basic element of L elements of vector p, the right sides μ_ℓ are also expressed in terms of the remaining elements of the vector. After manipulation the approximating expression takes the form

$$\lambda(T) = \sum_{k=1}^{m} p_k \psi_k(T), \tag{23}$$

where L elements of vector $\bar{p}$ are known, and $\psi_k(T)$, $k = \overline{1, m}$, is the transformed system of basic functions. If information on the values of sought derivative functions at a certain number of points of the interval [a, b] is available, it is accounted for in a similar way.

The algorithm presented is realized in the form of a package of computer programs. Here, relevant boundary-value problems are solved numerically using a finite-difference method. The program package was de-bugged and tested by solving model inverse problems numerically. The results confirmed the high efficiency of the proposed algorithm.

The algorithm devised for solving the inverse problem of heat conduction taking a priori information on values of the sought functions into consideration was utilized in processing and analyzing experimental data on the determination of the effective thermal conductivity for a composite material.

For the experimental study of the material heating processes, a radiation heating bench was used. The nonsteady heating process was controlled automatically by a designated surface temperature.

Parameters of the material heating were measured with the aid of specially designed sensors, allowing determination of the temperature as a function of time at several inner points of the material sample. Based on the analysis of existing methods to install thermocouples into the considered material, the current study adopted the following sensor design. The

TABLE 1. Temperature Dependence of Volumetric Thermal Conductivity of the Material

T, K	273	373	473	573
$C \cdot 10^{-6}$, J/(m^3·K)	0,825	1,125	0,75	0,9

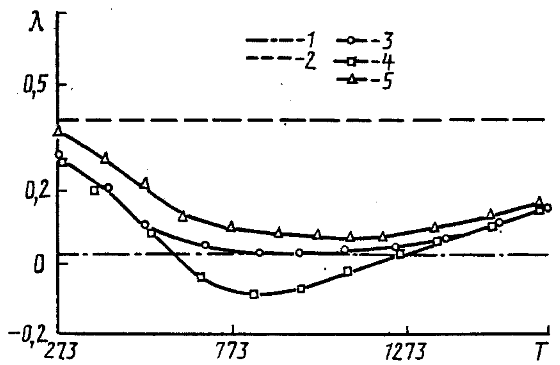

Fig. 1

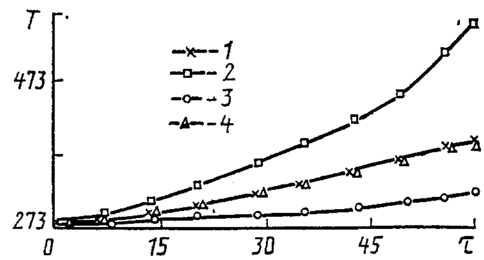

Fig. 2

Fig. 1. Determination of the thermal conductivity: 1) limitation; 2) initial approximation; 3, 4) calculated results on the 2-nd iteration before and after the projection; 5) calculated result on the 10-th iteration.

Fig. 2. Comparison between experimental and calculated temperatures: 1) experiment; 2, 3) calculated temperature on the 2-nd iteration before and after the projection; 4) calculated temperature on the 10-th iteration.

composite material sample was located on a metal substrate fabricated from an AMG-6 alloy of thickness 1 mm and dimensions 70 × 70 mm. The thickness of the studied material was as large as 5 mm. Butt-welded Π-shaped tungsten-rhenium thermocouples were mounted on the material surface through borings made preliminarily in the substrate. A Chromel-Alumel thermocouple was installed on the substrate surface.

The thermal conductivity was determined in the unilateral heating of the sample. The inner surface was thermally insulated with the help of a thermal-insulation material. Indications of the thermocouples mounted on the heated surface and the thermal insulation condition on the inner surface were employed as boundary conditions of the first and second kind, respectively. At the initial time the sample temperature was constant and equal to 289 K. Table 1 presents the specified temperature dependence of the volumetric thermal conductivity of the material.

The duration of the sample heating was 60 sec. Prior to processing the experimental data, computational experiments were carried out which revealed optimal parameters of the difference grid, viz., $n_\tau \times n_x = 50 \times 50$. It was also established that, for the examined experiments, it is expedient to approximate the unknown relation $\lambda(T)$ by the B-spline with "natural" boundary conditions ($\lambda''(a) = \lambda''(b) = 0$) and with four splitting intervals of the approximation region of the sought function ($m = 5$).

During the solution of the inverse problem [the initial approximation was taken to be equal to $\lambda^0 = 0.4$ W/(m·K)], in the 2nd iteration the sought function on the interval [621, 1182 K] assumed negative values, which is shown in Fig. 1. Here, the iteration process was terminated because the direct problem of heat conduction with negative coefficients cannot be solved. After the procedure of projecting the derived relation $\lambda(T)$ on the half-plane $\lambda \geq 0.02$ (see Fig. 1) has been accomplished, it is possible to continue the iteration process until the condition of halt from the residual, according to the iteration regularization principle [2], is fulfilled. The relevant calculated temperatures and the relation $\lambda(T)$ to be determined are also given in Figs. 2 and 1.

It should be noted that negative values of $\lambda(T)$ also appear during iterations with other initial approximations. Thus, for example, at $\lambda^0 = 0.15$ W/(m·K), the problem of negative values of $\lambda(T)$ arises on the 3rd iteration as well.

The investigations performed show that in some cases it is impossible to avoid the computation of negative values of $\lambda(T)$ in the iteration process regardless of initial approximations. The only way out here is taking into account the a priori information on the nonnegativeness of the characteristic to be determined. The proposed algorithm offers the prospects of the efficient solution for the inverse problem in situations of such kind.

NOTATION

T, temperature; τ, time; x, spatial coordinate; f, additional temperature measurements; C, volumetric specific heat; λ, thermal conductivity; q, external thermal effect; J, minimized functional; γ, descent step; δ^2, integral error of measurements; n, number of steps of the difference net.

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